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NAVAL POSTGRADUATE SCHOOL

Monterey, California

MODELING AND ESTIMATING

SYSTEM AVAILABILITY

by

Donald P. Gaver

January 1977

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A variety of probability models for single and multiple unit, failure-prone but repairable, systems are reviewed. The purpose of the paper is to provide methods for expressing the uncertainties in system availability in terms of uncertainties in component parameters. A log-linear transformation and the "jackknife" are shown to be effective.

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MODELING AND ESTIMATING SYSTEM AVAILABILITY

PART I. INTRODUCTION

The availability of an equipment, or system of components, such as an electric power generator or boiler, a nuclear reactor, or a reactor safety system, is defined as the probability that the system is "up", or able to perform its intended mission. Since equipments sometimes fail, system availability can be increased by scheduling inspections and allowing for preventive maintenance, and, when needed, corrective repairs, Also, the availability of a system is enhanced by the introduction of redundancy, i.e., by the use of parallel equipment.

The purpose of this report is to discuss the definition and measurement of availability from a statistical viewpoint. The statistical approach to problems of equipment reliability and availability begins by representing the individual component times between failures, and the subsequent down or repair times, by statistical variables having probability distributions. Aspects of this mathematical modeling step are described in Part II. There it is pointed out, for example, that long-run availability of individually maintained units depends only upon the mean or average time to failure, and the mean repair time of that, or similar, equipments.

Part III of this report considers the problem of the probable variability of availability from component to component, and its consequent effect upon system availability. For example, the mean

time between failures (MTBF) of a component of a particular type will vary because of manufacturing, environmental, and maintenance differences. There will be differences in the component availability as a consequence. If the variability of the MTBF, and also the mean time to repair (MTTR) of a component is represented by probability distributions, as applied in the Reactor Safety Report, WASH-1400, then statistical variability of the system availability is also implied. The problem considered in Part III is that of approximating the probability distributions of the availability of a system of, perhaps, many different components, given the probability distributions of individual component MTBF and MTTR. Having such a probability distribution it is possible to place a probability level on a prospective system of components (a reactor safety system, for instance) meeting a required availability.

In Part IV, it is shown how failure and repair data, amassed from experience with individual components, can be utilized to make statistical inferences about the true, but unknown, availability of the component. It is also shown how such data, available for each of many different components that make up a system, can be employed to infer system availability. The method used, called the jackknife, tends to be insensitive to the mathematical form of the underlying probability distribution of the times to failure and times to repair observed. This property is useful in practice since the latter distributions are unlikely to be known at all precisely. An example of the application of the jackknife technique to some actual failure and repair data obtained from the Humboldt Bay and

Yankee is presented in Part IV, Sec. 4.4. The confidence limits for the long-run availability of these two nuclear plants are calculated. The jackknife confidence limits are shown to resemble comparable limits obtained by two other methods, but actually to be slightly narrower than the latter.

The methodology of Parts III and IV are aimed at solving similar, but not identical problems. That of Part III addresses the problem of assessing the availability of a system of components <u>before</u> any data on the particular components is available. This is done on the basis of judgment or experience with similar components in and from different environments. The procedures of Part IV assess the availability of a particular system that is in operation, and whose components have been in operation, or under test, long enough to furnish some actual failure and repair data.

PART II. ANALYTICAL MODELS FOR AVAILABILITY

2.1 General

In this section several mathematical models are presented for the availability of a complex, repairable, and possibly redundant system. Relevant availability models are reviewed, and methods are suggested for obtaining numerical results from them, once having specified the probabilistic properties of components, such as the probability structure of the failure and repair processes. Suggestions are also made for obtaining time-dependent availability information from data on component failure and repair times.

2.2 Availability: One Element

Consider the time history of an item (e.g., a reactor safety system, an entire reactor, or a component of one of the above) that is in one of two states at any time: available, or unavailable. For short, say that the unit is "up" when available, and otherwise is "down". Suppose that the up time intervals, or times to failure, $\{U_i, i=1,2,..\}$, are a sequence of independent statistical variables, each having the distribution function $F(x)$; also suppose that the down time intervals $\{D_i\ i=1,2,..\}$ are likewise independently distributed with distribution function $G(y)$. Furthermore, if both $\{U_i\}$ and $\{D_i\}$ are statistically independent, then the random sequence (or <u>stochastic process</u>) $X(t)$ that takes on the value unity when the system is up, and zero when down, is called an <u>alternating renewal process</u> (see Cox [3]). Finally, $A(t)$,

the availability of the system at time t is defined to be

$$A(t) = \text{Probability the system is up at time } t$$
$$= P\{X(t) = 1 | X(0)\} \quad ,$$

where $X(0)$ refers to its condition at some initial time point, denoted by $t=0$. Explicit mathematical formulas for $A(t)$ will be derived and discussed; these naturally involve properties of the up time and down time distribution functions, F and G.

Note 1: Availability at <u>time t</u> depends upon initial conditions: whether the system is <u>up</u> at time $t=0$, perhaps immediately following repair, or down, immediately preceding repair. Thus, it is proper to define availability at time t, given the item state at $t=0$. For instance, the probability that the system is up at time t, given its initial state, written

$$A(t|X(0)) = P\{X(t) = 1|X(0)\}$$

is of interest: $X(0) = 1$ signifies that the item is up at $t = 0$. Under reasonable conditions $A(t|X(0))$ will tend to a constant, $A(\infty) = A$, as time increases. The latter <u>steady-state availability</u> is independent of the initial conditions. This measure of system effectiveness will be of principal concern in this report.

Note 2: Availability as described here, refers to the probability of item operability <u>at one point in time</u>, t. It may also be desirable to calculate an <u>interval availability</u>

$A(t,\Delta) = P\{X(t') = 1$ for all time t' between t and $t+\Delta\}$.

For instance, Δ is the time required for the item to complete its mission (which may be variable, and hence be modelled as a random variable).

Note 3: It may well be that there is interest in system availability <u>at demand</u>, and that demands, e.g., nuclear reactor accidents or earthquakes, etc., occur at variable times and can be treated as a random variable. For instance, let T be the random time at which a demand, or need, for the safety device occurs, therefore the demand availability is the mean value of the quantity A(T). It is sometimes easier to calculate this latter, more seemingly complex quantity than it is to calculate simple point availability.

Note 4: An item is in only one of two states in the present setup: available, or unavailable. We make no use of a concept of reduced operability at this stage, although such may indeed occur.

2.2.0 A Mathematical Model: General Independent Up and Down Times

Assume that $\{U_i\}$ are mutually independent and identically distributed with distribution function (d.f.) $F(x)$, and that $\{D_i\}$ have similar properties with d.f. $G(y)$. Assume also that the up and down times are mutually independent (a model without this latter assumption has been suggested and discussed by Gaver [2]).

2.2.1 Derivation of $A(t)$

Suppose that initially the system is just beginning an up time, and the availability at time t is to be calculated. Denote by $C = U_1 + D_1$, the time to complete exactly one failure-repair cycle. The time C has distribution function

$$F*G = \int_0^z F(z-y)dG(y) = P\{C \leq z\}, \qquad (2\text{-}2\text{-}1)$$

where * denotes the conventional convulation operation. The system is up at time t if either, (i) it is up at time t, never having failed, an event with probability $P[U_1 \geq t] = 1-F(t)$, or (ii) it has failed, been repaired, before t, and is up again at t. Expressed mathematically, this says that (we put $A_U(t)$ for availability, given it is up initially)

$$A_U(t) = 1-F(t)+\int_0^t A_U(t-z)\frac{d(F*G)}{dz}\, dz \qquad (2\text{-}2\text{-}2)$$

an integral equation for $A_U(t)$, given that the system was up initially. If the item is initially down the equation changes, but $A_D(t)$ is easily expressed in terms of $A_U(t)$:

$$A_D(t) = \int_0^t A_U(t-z)\,dG(z). \tag{2-2-3}$$

This expression simply says that the item is up at time t if it begins a down time at $t = 0$ which lasts until time z; then, starting in an up condition; as in (2-2-2), it is up at time t with probability $A_U(t-z)$; integrating over z gives (2-2-3).

2.2.2 Solution for A (t)

In general, a usable closed-form solution to the integral equations (2-2-2 and 3) is not available. One exception is notable, namely that in which up and down times are exponentially distributed. That is

$$F(x) = 1-e^{-\lambda x},$$
$$G(y) = 1-e^{-\mu y} \tag{2-2-4}$$

Equations (2-2-2) and (2-2-3) yield the fomulas

$$A_U(t) = e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}[1-e^{-(\lambda+\mu)t}], \tag{2-2-5}$$

and $$A_D(t) = \frac{\mu}{\lambda+\mu}[1-e^{-(\lambda+\mu)t}] \tag{2-2-6}$$

Note 1: If initially the item is up, then there is a decrease of availability until a steady-state value $\frac{\mu}{\lambda+\mu}$ is reached. Likewise, if the item is initially <u>down</u>, then the availability increases to $\frac{\mu}{\lambda+\mu}$. In both cases, the steady-state values are the same, and the approach is governed by the "time constant" $\lambda+\mu$.

Note 2: The steady-state availability is of the form

$$\lim_{t\to\infty} A_U(t) = \lim_{t\to\infty} A_D(t) = \frac{\mu}{\lambda+\mu} = \frac{E[U]}{E[U]+E[D]} \quad . \tag{2-2-7}$$

Thus, in the long run, the system availability is the average length of an up period divided by the average "cycle length", where "cycle" is defined to be an up period plus the following down period. The validity of equation (2-2-7) does not depend upon the properties of the distribution of U and D.

To find the general solution to equations (2-2-2) and (2-2-3) the Laplace transform technique may be used. If one takes Laplace tranforms in (2-2-2), the transform of the availability is found to be

$$A_U(s) = \frac{1}{s} \cdot \frac{1 - f(s)}{1-f(s)g(s)}, \tag{2-2-8}$$

see Reference [2].

where $$A_U(s) = \int_0^{\infty} e^{-st} A_U(t)\,dt \qquad (2\text{-}2\text{-}9)$$

$$f(s) = \int_0^{\infty} e^{-sx}\,dF(x),$$

$$g(s) = \int_0^{\infty} e^{-sy}\,dG(y).$$

In principle, the transform (2-2-8) provides the time-dependent solution desired. The inversion of the transformed equation (2-2-8) is sometimes difficult. Several "practical" remarks are in order.

Note 1: If $f(s)$ and $g(s)$ are both rational functions of s, e.g., if $g(s)$ and $f(s)$ are Erlang:

$$\frac{dF}{dx} = e^{-k\lambda x} \frac{(k\lambda x)^{k-1}}{(k-1)!} k\lambda, \qquad (2\text{-}2\text{-}10)$$

$$f(s) = \left(\frac{\lambda}{\lambda + \frac{s}{k}}\right)^k$$

or k a positive integer, and

$$\frac{dG}{dy} = e^{-j\mu y} \frac{(j\mu y)^{j-1}}{(j-1)!} j\mu, \qquad (2\text{-}2\text{-}11)$$

$$g(s) = \left(\frac{\mu}{\mu + \frac{s}{j}}\right)^j$$

again for j a positive integer, then explicit, but messy, mathematical inversion can be accomplished. Numerical results can be obtained by writing a FORTRAN program and even, very possibly, by use of a programable hand-held calculator. Since almost any distribution function can be well-represented by a distribution having rational Laplace transform, the above procedure can be carried out in practice.

Note 2: Computer programs have been developed for numerically inverting Laplace transforms, c.f. Gaver [1], and application of one of these is also practically possible. One must have the Laplace transforms of the component distribution functions of F and G in order to achieve the final result. In practice, again, one may well have observations from the latter: $u_1, u_2, \ldots, u_n$, and $d_1, d_2, \ldots, d_m$ ($n = m$, or $n \neq m$ for the sample sizes need not be the same). Now one can:

a) fit a plausible analytic form, e.g., a member of the gamma family, to F and G:

$$\hat{f}(s) = \left(\frac{\lambda}{\lambda + \frac{s}{\hat{j}}}\right)^{\hat{j}} \qquad (2\text{-}2\text{-}12)$$

$$\hat{g}(s) = \left(\frac{\mu}{\mu + \frac{s}{\hat{k}}}\right)^{\hat{k}},$$

and then apply a transform inversion routine. The parameter fits can be determined by maximum likelihood, or by the moment matching method, i.e., by equating the theoretical distribution's mean, variance, etc., to the corresponding mean and variance of the sample data, later solving for the distribution's parameters.

b) utilize the empirical Laplace transform, defined as

$$\hat{f}(s) = \frac{1}{n} \sum_{i=1}^{n} e^{-su_i}$$

$$\hat{g}(s) = \frac{1}{n} \sum_{i=1}^{n} e^{-sd_i} \tag{2-2-13}$$

and then apply a transform inversion routine to (2-2-8).

The actual operating characteristics of the above approaches--and variations thereof--remain to be evaluated. Very likely an experimental sampling or Monte Carlo approach will be required to shed light on their performance.

2.3 Availability: Several Elements

2.3.1 Steady State System Availability; Independent Elements.

System availability depends upon the availability of its subsystems and upon the operational logic. Suppose a system is composed of N elements where it is assumed that the up and down times (i.e., time-to-failure and repair time) of each element are statistically independent, then the system availability can be calculated from the element availability. Because of the independence assumption, this particular model may not be applicable to the common failure mode situation or to the situation of repairing of elements involving a waiting-queue (insufficient repairmen). Let A_i denote the steady-state availability of $i^{\underline{th}}$ element, then as in Eq. (2-2-7),

$$A_i = \frac{E[U_i]}{E[U_i]+E[D_i]}, \tag{2-3-1}$$

and the unavailability of the $i^{\underline{th}}$ element, $\bar{A}_i$, is given by

$$\bar{A}_i = 1-A_i.$$

The availability of several types of systems is derived below:

System Type 1. N Unit Redundant

If N elements are arranged in parallel, i.e., redundantly, so that the system operates if, and only if, at least one operates,

then system unavailability is on the basis of element independence,

$$\bar{A} = \bar{A}_1 \cdot \bar{A}_2 \ldots \bar{A}_N = \prod_{i=i}^{N} \bar{A}_i \qquad (2\text{-}3\text{-}2)$$

or, equivalently, availability is

$$A = 1 - (1-A_1)\ (1-A_2) \ldots (1-A_N) = 1-\bar{A} \qquad (2\text{-}3\text{-}3)$$

System Type 2. M out of N Unit Redundant

If N items are arranged in a system so that if at least M operates ($1 \leq M \leq N$), the system operates, then system availability can be computed (again using the independence assumption) as follows:

(a) Compute the probability that each set of exactly M units operates (the remaining set of $N - M$ does not operate). There are $\binom{N}{m} = \frac{N!}{m!\,(N-m)!}$ such sets. Add these individual probabilities.

(b) Add the probabilities of (a) for $m = M, M+1, \ldots N$. This is the required result.

As an illustration, consider the two out of three system; here $M = 2$, $N = 3$. The results of steps (a) and (b) are as follows:

(a) $m = 2$: $\bar{A}_1\ A_2\ A_3 + A_1\ \bar{A}_2\ A_3 + A_1\ A_2\ \bar{A}_3$

$m = 3$: $A_1\ A_2\ A_3$ (2-3-4)

(b) A = system availability

$= A_1\ A_2\ A_3 + \bar{A}_1\ A_2\ A_3 + A_1\ \bar{A}_2\ A_3 + A_1\ A_2\ \bar{A}_3$

A recursive scheme to calculate system availability is now described.

Procedure

(1) Enumerate the elements, the $i^{\underline{th}}$ being called Element i, $i = 1,2,\ldots,N$.

(2) Define

(a) $a(j,k)$ = Probability that exactly j out of the first k elements are up $(0 \le j \le k)$.

(b) $A(M,N)$ = Availability of a M out of N system

$$= \sum_{j=M}^{N} a(j,N) \tag{2-3-5}$$

(c) Compute $a(j,k)$ for $j \le k \le N$

$$a(j,k) = a(j,\ k-1)\bar{A}_k + a(j-1,\ k-1)A_k \tag{2-3-6}$$

to obtain $a(j,k)$, $M \le j \le N$;

and where

$$\begin{aligned} A(0,1) &= a(0,1) = \bar{A}_1 \\ A(1,1) &= a(1,1) = A_1 \end{aligned} \tag{2-3-7}$$

and also

$$A(M,M) = a(M,M) = \prod_{i=1}^{M} A_i \qquad (2\text{-}3\text{-}8)$$

(d) Compute

$$A(M,N) = \sum_{j=M}^{N} a(j,N)$$

This is the required availability.

In order to explain the recursive formula (2-3-6) notice that j out of the first k elements are available if either j out of the first $k-1$ are available and the $k^{\underline{th}}$ is unavailable, or if $j-1$ out of the first $k-1$ are available, and the $k^{\underline{th}}$ is available.

A return to the previous example illustrates the technique.

First,

$$\begin{aligned} a(1,2) &= a(1,1)\,\bar{A}_2 + a(0,1)\,A_2 \\ &= A_1\,\bar{A}_2 + \bar{A}_1\,A_2 \end{aligned} \qquad (2\text{-}3\text{-}9)$$

Next, using (2-3-8) and also (2-3-9),

$$\begin{aligned} a(2,3) &= a(2,2)\,\bar{A}_3 + a(1,2)\,A_3 \\ &= A_1\,A_2\,\bar{A}_3 + (A_1\,\bar{A}_2 + \bar{A}_1\,A_2)\,A_3 \end{aligned} \qquad (2\text{-}3\text{-}10)$$

Since, $a(3,3) = A_1\,A_2\,A_3$ according to (2-3-8), this added to (2-3-10) delivers the required result, by (2-3-5).

System Type 3. M out of N Unit Redundant, Identically Available Units.

This is the same system logic as immediately above. But since the units are believed to have equal availabilities, the binomial distribution can be used to calculate system availability from component availability:

$$A = \sum_{i=M}^{N} \binom{N}{i} A_0^{\,i} (1-A_0)^{N-i} \qquad (2\text{-}3\text{-}11)$$

Here A_0 denotes the i^{th} individual system availability. The Equation (2-3-11) has been extensively tabled, and so is convenient to use, if appropriate.

2.3.2 More Complex Models; Transients, Dependence

In order to deal with more complex models of system availability it is useful to use Markov process models; (see Gaver and Thompson [4] or Karlin [5] for an introduction). Only a brief discussion will be given here, and that in terms of examples.

Example 1. Single Unit

Consider a single system element or unit, with failure rate at time t being $\lambda(t)$, and repair rate $\mu(t)$. The time dependence of these rates may be used to represent reliability growth: $\lambda(t)$ may well decrease with time because initial difficulties are found and removed, and $\mu(t)$ may increase because of greater familiarity with the system on the part of those responsible for its maintenance.

Let $P_0(t)$ be the probability that the unit is up at time t, and $P_1(t) = 1-P_0(t)$ be the probability that it is down for repair. Then the probability that the unit is up at time $t + h$ can be written as follows:

$$P_0(t+h) = P_0(t)[1-\lambda(t)h] + P_1(t)\mu(t)h + R(t,h) \qquad (2\text{-}3\text{-}12)$$

In other words, Equation (2-3-12) states that the unit is up at $t+h$ $(h > 0)$ if (i) it is up at t (probability $P_0(t)$) and does not fail during the time from t to $t+h$ with probability approximately $1-\lambda(t)h$, or (ii) it is down at t with probability $P_1(t)$ and is repaired between t and $t+h$ (probability $\mu(t)h$). Other possibilities have the probability $R(t,h)$, which according to the Markov assumption is small compared to h (literally, the limit of $R(t,h)/h$ as h tends to zero is zero). Note that neither the time since last failure, nor the time that repair has been going on, influences the probability of state change. This is the "Markov property".

Now subtract $P_0(t)$ from both sides of Equation (2-3-12), and divide by h; let h tend to zero. We have then the following differential equation,

$$\begin{aligned} \frac{dP_0}{dt} &= -\lambda(t)P_0(t) + \mu(t)\,P_1(t) \qquad (2\text{-}3\text{-}13) \\ &= -\left[\lambda(t) + \mu(t)\right] P_0(t) + \mu(t) \end{aligned}$$

The solution may be expressed as

$$P_0(t) = P_0(o)\,e^{-r(t)} + \int_o^t e^{-r(t-z)}\,\mu(z)\,dz$$

where $r(t) = \int_0^t [\lambda(x) + \mu(x)]dx$, and $P_0(0)$ is the probability that the system is up at time $t=0$. If $\lambda(t) = \lambda$, $\mu(t) = \mu$ are constants, then

$$P_0(t) = P_0(o)\, e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}\left[1-e^{-(\lambda+\mu)t}\right] \tag{2-3-15}$$

so that if the system is up initially $P_0(0) = 1$,

$$P_0(t) = \frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu} \tag{2-3-16}$$

while if it is down for repair initially $P_0(0) = 0$,

$$P_0(t) = \frac{\mu}{\lambda+\mu}\left[1-e^{-(\lambda+\mu)t}\right] \tag{2-3-17}$$

It may be observed that the expressions (2-3-16) and (2-3-17) describe the effect of initial conditions on availability at time t, as described in Section 2.2.2. As time $t\to\infty$ in either expression, $P_0(t)$ -- which is equal to $A(t)$, the probability that the unit is available -- approaches A, the steady-state expression (2-2-3), by virtue of the fact that $E[U] = \lambda^{-1}$, and $E[D] = \mu^{-1}$.

Example 2. Three Units

If there are several units, then the <u>system state</u> must describe which are up. For instance,

$P_{0,0,0}(t)$ = Probability Units 1, 2, 3 are up at t.

$P_{1,0,0}(t)$ = Probability Unit 1 is Down, Units 2 and 3 are Up at t.

.
.
.

$P_{1,1,1}(t)$ = Probability Units 1, 2, 3 are Down at t.

There are, in all, eight states: (0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), 1,0,1), (0,1,1), (1,1,1), and their associated probabilities, for which differential equations may be written. Thus by the same argument as utilized to derive equations (2-3-13), the system of equations, (the parameters can be time-dependent), are given below:

$$\frac{d}{dt} P_{000}(t) = -(\lambda_1+\lambda_2+\lambda_3)P_{000}(t) + \mu_1 P_{100}(t) + \mu_2 P_{010}(t) + \mu_3 P_{001}(t) \quad (2\text{-}3\text{-}18)$$

$$\frac{d}{dt} P_{100}(t) = -(\mu_1+\lambda_2+\lambda_3)\, P_{100}(t) + \lambda_1\, P_{000}(t) + \mu_2\, P_{110}(t) + \mu_3\, P_{101}(t) \quad (2\text{-}3\text{-}19)$$

$$\frac{d}{dt} P_{111}(t) = -\,(\mu_1+\mu_2+\mu_3)\, P_{111}(t) + \lambda_1\, P_{011}(t) + \lambda_2\, P_{101}(t) + \lambda_3\, P_{110}(t) \quad (2\text{-}3\text{-}20)$$

For the setup above it turns out that, since all units are independent, the solution can be expressed as products of solution c single-unit problems, i.e., using equations (2-3-15) to (2-3-18) s appropriate.

The differential equation approach can be used to model systems in which component availabilities are not independent, perhaps because of limited repair capability. Suppose, for instance, that there is only one repairman, and that he assigns priority to units 1, 2, 3 in that order if the elements are down. In other words, if the repairman is repairing Unit 2, and if Unit 1 fails, he immediately changes to Unit 1. In this case, equation (2-3-18) remains the same but equation (2-3-19) becomes

$$\frac{d}{dt} P_{100}(t) = -(\mu_1+\lambda_2+\lambda_3)\, P_{100}(t) + \lambda_1\, P_{000}(t), \qquad (2\text{-}3\text{-}21)$$

and

$$\frac{d}{dt} P_{110}(t) = -(\mu_1+\lambda_3)\, P_{110}(t) + \lambda_1\, P_{010}(t) + \lambda_2\, P_{100}(t)$$

but

$$\frac{d}{dt} P_{010}(t) = -(\lambda_1+\mu_2+\lambda_3)\, P_{010}(t) + \lambda_2\, P_{000}(t) + \mu_1\, P_{110}(t) \qquad (2\text{-}3\text{-}22)$$

and, finally,

$$\frac{d}{dt} P_{111}(t) = -\mu_1\, P_{111}(t) + \lambda_1\, P_{011}(t) + \lambda_2\, P_{101}(t) + \lambda_3\, P_{110}(t) \qquad (2\text{-}3\text{-}23)$$

The long-run or steady-state probabilities are derived by equating the derivatives to zero, and solving the resulting system of linear equations, subject to the condition that the sum of the probabilities equals one. It is recommended that a computer routine be used for this, as the explicit solution is very messy. The time-dependent or transient solution may also be obtained by numerically integrating the differential equations; a Runge-Kutta method will work well.

Finally, the availability can be calculated in an obvious way from the probabilities as obtained. For instance if the system logic requires that <u>at least one be operative</u>, then

$$A(t) = 1 - P_{111}(t) \tag{2-3-24}$$

while, if two out of three operative is required, then

$$A(t) = P_{000}(t) + P_{100}(t) + P_{010}(t) + P_{001}(t) \tag{2-3-25}$$

more complex setups, including common mode failures, may be treated similarly.

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PART III. APPROXIMATE CONFIDENCE LIMITS FOR SYSTEM AVAILABILITY

3.1. General

The steady state system availability of a complex system depends upon the availabilities of its components and the system operational logic. Denote the system availability, A_s,

$$A_s = \phi(A_1, A_2, \ldots, A_n), \tag{3-1-1}$$

where A_i is the steady state availability of the $i^{\underline{th}}$ component and the function ϕ is a system logic function, which describes system availability in terms of component availability.

Furthermore, under broad circumstances, and as a first approximation,

$$A_i = \frac{E[U_i]}{E[U_i]+E[D_i]} \tag{3-1-2}$$

where $E[U_i]$ represents the expected up time or time to failure and $E[D_i]$ is the expected down time of the $i^{\underline{th}}$ subsystem. Now judgments about, and experience with, the component availabilities, A_i, will differ, and so it may be natural and useful to represent this variability by probability distributions (somewhat in the spirit of Bayesian statistics, see DeGroot [2]). In fact, the Reactor Safety Report, WASH-1400 has adopted this notion; specifically, it assumes the logarithmic-normal distribution to

describe the variability of $E[U_i]$, and $E[D_i]$, or equivalently $\mu_1 = (E[U_i])^{-1}$ and $\lambda_i = (E[D_i])^{-1}$, respectively. That is, the availabilities of similar components of type i vary randomly; therefore,

$$A_i = \frac{\underset{\sim}{\mu}_i}{\underset{\sim}{\lambda}_i + \underset{\sim}{\mu}_i} \tag{3-1-3}$$

is a statistical variable, where $\ell n\ \underset{\sim}{\lambda}_i$ is Normal $(m_{\lambda_i}, \sigma^2_{\lambda_i})$ and $\ell n\ \underset{\sim}{\mu}_i$ is Normal $(m_{\mu_i}, \sigma^2_{\mu_i})$. Consequently the availability of a system constructed of such elements is also a random variable. The problem is to assign a probability number to the event that the availability of a system exceeds a given lower bound, given the distributions of component failure and repair rates. Equivalently, one can specify a lower bound, $\underline{\alpha}_s$, such that system availability $\underset{\sim}{A}_s$, exceeds it with a specified probability.

Under the assumptions made, the problem cannot be solved in a neat, closed-form, manner. This part of the report proposes an approximation method which provides a satisfactory approximation (as indicated by a Monte Carlo simulation study). However, further investigations are recommended. The method is known as Linearizing System Availability Log-Odds (abbreviated LALOD).

3.2. Linearizing Availability Log-Odds: Rationale

3.2.1 Single Component System.

Consider first a system consisting of a single component, and express its availability in the following equivalent forms:

$$A_s \equiv A = \frac{\mu}{\lambda + \mu} = \frac{1}{1 + \frac{\lambda}{\mu}}$$

$$= \frac{1}{1 + e^{[\ell n\lambda - \ell n\mu]}} . \qquad (3\text{-}2\text{-}1)$$

The parameters λ and μ are realizations of random variables $\underset{\sim}{\lambda}$ and $\underset{\sim}{\mu}$. Let L_s be the LALOD variable and

$$L_s \equiv \ell n\underset{\sim}{\lambda} - \ell n\underset{\sim}{\mu} \qquad (3\text{-}2\text{-}2)$$

In the WASH-1400 case where $\underset{\sim}{\lambda}$ and $\underset{\sim}{\mu}$ are log-normally distributed L_s would be a normally distributed ramdom variable with mean $m = m_\lambda - m_\mu$ and variance $\sigma^2 = \sigma^2_\lambda + \sigma^2_\mu$. Furthermore, the LALOD variable, L_s, can be expressed as a function of system availability,

$$L_s = \ell n \left[\frac{1-\underset{\sim}{A}_s}{\underset{\sim}{A}_s}\right]; \qquad (3\text{-}2\text{-}3)$$

thus, the distribution function of $\underset{\sim}{A}_s$ is given by

$$P\left\{A_s > \alpha_s\right\} = P\left\{L_s \leq \ell n \left[\frac{1-\alpha_s}{\alpha_s}\right]\right\}$$

$$P\left\{A_s > \underline{\alpha}_s\right\} = P\left\{\varepsilon \leq \sigma^{-1}\left[\ell n\left((1-\underline{\alpha}_s)\underline{\alpha}_s^{-1}\right) - m\right]\right\} \tag{3-2-4}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{b_s} e^{-\frac{1}{2}u^2}\,du$$

where ε has a standard normal distribution (with mean zero and variance unity) and

$$b_s = \sigma^{-1}\left[\ell n\left((1-\underline{\alpha}_s)\ \underline{\alpha}_s^{-1}\right) - m\right] \tag{3-2-5}$$

To construct the one-sided probability limit of $\underset{\sim}{A}_s$ for a given level of significance, α , equation (3-2-5) can be used to determine an $\underline{\alpha}_s$ value since b_s can be found from the standard normal probability table for that given value of α.

Note 1: The distribution function and the probability limit derived for the system availability $\underset{\sim}{A}_s$ is exact under the assumption that $\ell n(\underset{\sim}{\lambda}/\underset{\sim}{\mu})$ is a normal random variable.

NOTE 2: The assumption that $\ell n(\underset{\sim}{\lambda}/\underset{\sim}{\mu})$ is Normal is not the only possibility: under some circumstances another transformation may be more suitable. In fact, a transformation to another basic distribution, other than the Normal, may be indicated by data. In any case, the odds transformation is still helpful numerically. This

particular transformation has been systematically explored by Cox in a data-analytical context, see Cox [1]. The same arguments that make it appealing in that context tend to recommend it for the present purposes.

Note 3: The log-odds transformation in equation (3-2-3) has range $-\infty < L_s < \infty$, corresponding to the domain $0 < A_s < 1$: $A_s = 0$ corresponds to $L_s = \infty$, and $A_s = 1$ corresponds to $L_s = -\infty$. It is immaterial whether L_s be defined as shown, or as the log of the inverted ratio. In any case, L_s ranges over the natural region of definition of the normal distribution, and will be more nearly normally distributed than will A_s itself.

3.2.2. Multiple Unit System

Now consider a system consisting of several units arranged in a redundant manner. The general procedure of LALOD transformation is outlined below:

LALOD Procedure

(1) Form the system availability in terms of component availabilities:

$$A_s = \phi(A_1, \ldots, A_m).$$

(2) Form the log-odds availability, L_s,

$$L_s = \ln\left[\frac{1-A_s}{A_s}\right] = \ln\left[\frac{1 - \phi(A_1, A_2, \ldots, A_n)}{\phi(A_1, A_2, \ldots, A_n)}\right] \tag{3-2-6}$$

(3) Compute the center of the log-odds distribution:

$$a_s = \phi(a_1, a_2, \ldots a_m) \tag{3-2-7}$$

where

$$a_i = \frac{1}{1 + e^{m_i}} \tag{3-2-8}$$

$$m_i = m_{\lambda_i} - m_{\mu_i} = E[\ln \underset{\sim}{\lambda_i}] - E[\ln \underset{\sim}{\mu_i}].$$

(4) Compute the linearized approximation to the variance of log odds availability by use of the formula

$$\sigma_s^2 = \frac{1}{[a_s(1-a_s]^2} \sum_{i=1}^{m} \left(\frac{\partial \phi}{\partial a_i}\right)^2 a_i^2 (1-a_i)^2 \sigma_i^2 \tag{3-2-9}$$

(5) Express the system log odds availability as

$$L_s \approx \ln \left[\frac{1-a_s}{a_s}\right] + \varepsilon \, \sigma_s \equiv Z_s \tag{3-2-10}$$

where ε is Normal (0,1). Thus, by using equation (3-2-4), the following approximation is obtained for the probability that the availability fo a system exceeds a lower bound $\underline{\alpha}_s$:

$$P\left\{A_s \geq \underline{\alpha}_s\right\} = P\left\{\varepsilon < \left(\ell n\left[\frac{1-\underline{\alpha}_s}{\underline{\alpha}_s}\right] - \ell n\left[\frac{1-a_s}{a_s}\right]\right)\sigma_s^{-1}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{b_s} e^{-\frac{1}{2}u^2}\, du \tag{3-2-11}$$

$$b_s = \sigma_s^{-1}\ \ell n\left[\left(\frac{1-\underline{\alpha}_s}{1-a_s}\right)\left(\frac{a_s}{\underline{\alpha}_s}\right)\right]$$

From equation (3-2-11) one can easily determine desired probability limits. To determine $\underline{\alpha}_{s,p}$ such that

$$P\{\underset{\sim}{A}_s \geq \underline{\alpha}_{s,p}\} = p, \tag{3-2-12}$$

simply compute

$$\underline{\alpha}_{s,p} = \frac{a_s}{a_s + (1-a_s)\, e^{\sigma_s \varepsilon_p}} \tag{3-2-13}$$

where ε_p is the $p^{\underline{th}}$ quantile of the unit normal;

$$p = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\varepsilon_p} e^{-\frac{1}{2}u^2}\, du, \tag{3-2-14}$$

available from tables of the Normal distribution.

Note 1: The a_s of Equation (3-2-7) is precisely the mean or expected value of the log-odds availability for a <u>single</u> unit. The transformation tends to symmetrize A_s; A_s approximates the mean or center of the L_s distribution when a system involves more than one unit.

Note 2: Derivation of Equation (3-2-9) can be accomplished by first writing the differential

$$dL_s = -\frac{1}{\phi(1-\phi)} \sum_{i=1}^{m} \frac{\partial\phi}{\partial A_i} dA_i \tag{3-2-15}$$

and then differentiating $\ell n \dfrac{1-A_i}{A_i} = Z_i$ to express the local variation of L_s near its center as

$$dL_s \simeq \frac{1}{a_s(1-a_s)} \left[\sum_{i=1}^{m} \frac{\partial\phi}{\partial a_i} a_i (1-a_i) \sigma_i \epsilon_i \right]. \tag{3-2-16}$$

Squaring and taking expectations results in the variance equation (3-2-9). The same basic procedure can be extended to handle correlations between units.

To demonstrate the application of LALOD approximation, an example is given below:

<u>Example: Two-Component Redundant System</u>

Consider a system which consists of two parallel redundant units; the operation logic is assumed to be one-out-of-two. Thus, the system unavailability or availability is given by

$$\bar{A}_s = \bar{A}_1\bar{A}_2 \tag{3-2-17}$$

or

$$A_s = 1 - (1-A_1)(1-A_2) = \phi(A_1, A_2). \tag{3-2-18}$$

Thus step (1) gives

$$L_s = \ell n \left[\frac{\bar{A}_1 \bar{A}_2}{1-\bar{A}_1 \bar{A}_2} \right] \tag{3-2-19}$$

Step (2) yields

$$a_s = 1 - \left[1 + e^{-m_1} \right]^{-1} \left[1 + e^{-m_2} \right]^{-1}$$

from which the center of L_s is found to be

$$E\,[L_s] = \ell n \left[\frac{(1+e^{-m_1})^{-1}\ (1+e^{-m_2})^{-1}}{1 - (1+e^{-m_1})^{-1}\ (1+e^{-m_2})^{-1}} \right] \tag{3-2-20}$$

$$= m_1 + m_2 - \ell n[1 + e^{+m_1} + e^{m_2}].$$

If $\underset{\sim}{\lambda}_i << \underset{\sim}{\mu}_i$, as is likely, then m_i is negative and in magnitude around -3 to -6. Hence the center of the L_s distribution is likely to be near $m_1 + m_2$.

Next, step (3) approximates $\mathrm{Var}[L_s]$ by

$$\sigma_s^2 = \frac{1}{(1-\bar{a}_1 \bar{a}_2)} [a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2], \tag{3-2-21}$$

which can be expressed in terms of m_i. Once again if $\underset{\sim}{\lambda}_i << \underset{\sim}{\mu}_i$ then, for even moderately negative m_i, $\sigma_s^2 \approx \sigma_1^2 + \sigma_2^2$.

In passing, note that the general n-component sytem is equally easy to approximate in the manner described. For this

$$\bar{a}_s = \prod_{i=1}^{n} \bar{a}_i \qquad (3\text{-}2\text{-}22)$$

$$E\,[L_s] = \ln \left[\frac{1-a_s}{a_s}\right] \qquad (3\text{-}2\text{-}23)$$

$$\sigma_s^2 = \frac{1}{(1-\bar{a}_s)^2} \sum_{i=1}^{n} a_i \sigma_i^2 \qquad (3\text{-}2\text{-}24)$$

3.3. Some Simulation Validations

Monte Carlo simulation is used to validate the adequacy of the proposed LALDO approximation. To do so, realizations of component availabilities are obtained as follows,

$$A_i = \frac{1}{1 + e^{m_i + \varepsilon_i \sigma_i}} \qquad (3\text{-}3\text{-}1)$$

where m_i and σ_i are given, and where ε_i represents a random normal number with mean zero and standard deviation unity. The system availability is then calculated according to the system logic function ϕ at the values of A_i. Identify each realization A_i so obtained and use equation (3-2-13) to obtain $\underline{\alpha}_{s,p}$. Finally, compare the fraction of say, $n = 1000$ repetitions that fall

above $\alpha_{-s,p}$ with the approximated probability p. If the fraction agrees with p to within sampling error, the approximation method is, therefore, desirable.

Several such sampling validations are performed. The results as shown in Table 3.1 are in good agreement. A detailed explanation of the simulation runs follows. Recall that if the statistical variable X has the log normal distribution, i.e., $\ell n\ X \sim N(m,\sigma^2)$, then

$$E\ [X] = e^{m+\sigma^2/2} \tag{3-3-2}$$

$$\mathrm{Var}\ [X] = e^{2m+\sigma^2}\ [e^{\sigma^2} -1]$$

and the coefficient of variation

$$C(X) = \mathrm{Var}\ [X] \div (E[X])^2 = e^{\sigma^2} -1$$

For the first case in the table (3-1), a choice of $m_1 = \ell n(10^{-3}/2)$ and $\sigma_1^2 = \ell n4$ for the population from which Component I was selected (the mean failure rate from that population is 10^{-3} (days), with coefficient of variation of 3). Component II was selected at random from a population having mean failure rate 0.5×10^{-3}. In all cases, the repair time was assumed to be exactly one day in duration, merely to simplify the sampling experiment. Next, the lower limit on system availability $\underline{\alpha}_s(p)$ was computed, using equation (3-2-13) with the above parameters and a particular value of p. A total of 1000 redundant systems were then simulated,

and the fraction whose availability exceed $\alpha_{-s}(p)$ was obtained. It is these fractions that appear in the body of the table; for instance, in the first case 0.503 corresponds to $p = 0.5$, 0.790 to $p = 0.80$, and 0.959 to $p = 0.95$.

The computer program utilized to produce the quoted results will also simulate more complex redundant systems.

3.4 Conclusions

The LALOD procedure for constructing probability (Bayes prior) limits on system availability is computational simple. Based on the simulation results to date, the method appears to be valid. Further validation experiments, and analytical investigations of the method, would seem to be indicated.

Two related general areas for further investigation are the following:

(a) The robustness or insensitivity of the LALOD method to the specific assumption of the log normal for unit parameter priors. There are indications that the procedure may be relatively insensitive, particularly when used to evaluate rather complicated redundant systems, by virtue of central limit theorem effects.

(b) The possibility of combining the LALOD prior approach with data to form a posterior, in the strict Bayesian sense. Perhaps better, another method for "borrowing strength" from experience with other units in other locations can be devised. Also, the approximate normality of the system log odds may be exploited to yield a useful sequential procedure for assessing system availability.

SAMPLE VALIDATIONS

Table 3.1

Two-Unit Redundant System
(1000 repetitions per case)

Cases	$p = 0.5$	$p = 0.80$	$p = 0.95$
$m_1 = \ln(10^{-3}/2)$, $\sigma_1^2 = \ln 4$ $m_2 = \ln(10^{-3}/4)$, $\sigma_2^2 = \sigma_1^2$ $E[\underset{\sim}{\lambda}_1] = 10^{-3}$, $E[\underset{\sim}{\lambda}_2] = 10^{-3}/2$ $CV[\underset{\sim}{\lambda}_1] = CV[\underset{\sim}{\lambda}_2] = 3$	0.503	0.790	0.959
$m_1 = \ln(10^{-1}/\sqrt{2})$, $\sigma_1^2 = \ln 2$ $m_2 = \ln(10^{-3}/4)$, $\sigma_2^2 = \ln 4$ $E[\underset{\sim}{\lambda}_1] = 10^{-1}$, $E[\underset{\sim}{\lambda}_2] = 10^{-3}/2$ $CV[\underset{\sim}{\lambda}_1] = 1$, $CV[\underset{\sim}{\lambda}_2] = 3$	0.531	0.835	0.962
$m_1 = \ln(10^{-1}/\sqrt{2})$, $\sigma_1^2 = \ln 2$ $m_2 = \ln(10^{-3}/\sqrt{2})$, $\sigma_2^2 = \ln 2$ $E[\underset{\sim}{\lambda}_1] = 10^{1}$, $E[\underset{\sim}{\lambda}_2] = 10^{-3}$ $CV[\underset{\sim}{\lambda}_1] = 1$, $CV[\underset{\sim}{\lambda}_2] = 1$	0.483	0.805	0.955

Note: For simplicity only, $E[\mu_i] = 1$ and $\sigma_{\mu_i}^2 = 0$ throughout the above. Also, $\sigma_i^2 \equiv \sigma_{\lambda_i}^2$, and CV(.) stands for coefficient of variation.

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PART IV. AVAILABILITY ESTIMATION BY USE OF THE JACKKNIFE

4.1. General

Consider now the problem of estimating the availability of a single equipment from data on its up and down times: $u_1, u_2, \ldots u_n$, and $d_1, d_2, \ldots d_n$, respectively. By virtue of equation (2-2-7), namely,

$$A_i = \frac{E[U_i]}{E[U_i] + E[D_i]} \tag{4-1-1}$$

one could estimate A_i as follows:

$$A = \frac{\bar{u}}{\bar{u} + \bar{d}} \tag{4-1-2}$$

where as usual the bars denote averages:

$$\bar{u} = \frac{1}{n} \sum_{i=1}^{n} u_i \tag{4-1-3}$$

$$\bar{d} = \frac{1}{n} \sum_{i=1}^{n} d_i$$

However, because $\bar{u}$ and $\bar{d}$ are only approximations to the true means the resulting approximations for A can be quite poor.

In practice it will be of interest to estimate the availability of a single equipment, e.g., a power plant, or a redundant combination of equipments, such as a safety device, by

using observed time-to-failure and down time data. Also, an assessment of the stability of the estimates, perhaps in the form of confidence limits, will be desirable. Such a program can, in principle, be carried out by (i) postulating distributional forms for the up or failure times, U, and down or repair times, D, (ii) fitting the parameters of the latter distributions according to satisfactory statistical procedures, such as maximum likelihood or, possibly, Bayesian techniques, and (iii) substituting the parameter estimates into the availability formulas, such as equation (4-1-2). In order to find confidence limits, a linearization technique that relies on the asymptotic normality of maximum likelihood estimates may be employed.

This paper presents a procedure alternative to the above; it has been called <u>the jackknife</u> by J.W. Tukey. For further discussion see Mosteller and Tukey [12], also Cox and Hinkley [4], and Gray and Schucany [10]; a review has recently been furnished by R.G. Miller [11]. In brief, the jackknife method has the capacity to reduce the bias of estimates of such quantities as system availability, and also to furnish confidence limits that behave in a satisfactory manner--economically enclose the true availaiblity-despite the fact that underlying distributions are unknown. Demonstration of these properties can be carried out mathematically when sample sizes are large, but in realistic situations the jackknife technique must be validated by Monte Carlo simulation. A number of such simulation results are presented in this paper, and comparison with alternative methods are given.

4.2. A Jackknife Procedure for a Single Unit.

Jackknifed estimates and confidence limits are constructed by successively leaving out parts of the available data to construct pseudovalues. These are then averaged, and the stability of the average assessed by Student's - t in order to obtain confidence limits. The procedure is given as below:

(1) Transform first (see Mosteller and Tukey [12]) estimated 2-2-7:

$$\ln \frac{\tilde{A}}{1-\tilde{A}} = \ln \bar{u} - \ln \bar{d}; \tag{4-2-1}$$

jackknifing will be carried out using the statistic $\ln \bar{u} - \ln \bar{d} = z$.

(2) Recompute z repeatedly, leaving out successively the sample pairs $(u_1,d_1),(u_2,d_2),\ldots(u_j,d_j),\ldots(u_n,d_n)$

$$z_{-j} = \ln\left[\sum_{i=1}^{j-1} u_i + \sum_{i=j+1}^{n} u_i\right] - \ln\left[\sum_{i=1}^{j-1} d_i + \sum_{i=j+1}^{n} d_i\right] \tag{4-2-2}$$

$j = 1,2,\ldots n.$

(3) Compute the <u>pseudovalues</u> as follows:

$$z_j = nz - (n-1)z_{-j} \qquad j=1,2,\ldots,n; \tag{4-2-3}$$

recall that $z \equiv z_{all}$ is the result of computing the quantity to be jackknifed, leaving out none of the data.

(4) Compute the mean and variance of the pseudovalues:

$$\bar{z} = \frac{1}{n} \sum_{j=1}^{n} z_j \tag{4-2-4}$$

$$s_z^2 = \frac{1}{n-1} \sum_{j=1}^{n} (z_j - \bar{z})^2$$

(5) The jackknifed point estimate of the availability is now

$$\tilde{A}_{jk} = \frac{e^{\bar{z}}}{1+e^{\bar{z}}} \tag{4-2-5}$$

(6) "Symmetric" two-sided confidence limits at confidence level $(1-\alpha)100\%$ are derived as follows:

$$\bar{z} + t_{1-\alpha/2}(n-1) \sqrt{\frac{s_z^2}{n}} = H_\alpha \tag{4-2-6}$$

$$\bar{z} - t_{1-\alpha/2}(n-1) \sqrt{\frac{s_z^2}{n}} = L_\alpha$$

where $t_{1-\alpha/2}(n-1)$ is the $(1-\frac{\alpha}{2})100\%$ quantile of Student's - t with n-1 degrees of freedom. Then

$$\frac{e^{L_\alpha}}{1+e^{L_\alpha}} \le A \le \frac{e^{H_\alpha}}{1+e^{H_\alpha}} \tag{4-2-7}$$

with confidence approximately $(1-\alpha)100\%$. Note that the confidence limits are nearly symmetric around $\ell n(E[U]/E[D])$, and not around A.

(7) One-sided confidence limits at confidence level $(1-\alpha)100\%$ are derived as follows

$$\bar{z} + t_{1-\alpha}(n-1)\sqrt{\frac{s_z^2}{n}} = H_\alpha \qquad (4\text{-}2\text{-}8)$$

$$\bar{z} - t_{1-\alpha}(n-1)\sqrt{\frac{s_z^2}{n}} = L_\alpha$$

so a one-sided <u>upper</u> confidence limit is

$$A \leq \frac{e^{H_\alpha}}{1+e^{H_\alpha}} \qquad (4\text{-}2\text{-}9)$$

and a <u>lower</u> confidence limit is

$$A \geq \frac{e^{L_\alpha}}{1+e^{H_\alpha}} \qquad (4\text{-}2\text{-}10)$$

both at confidence level $(1-\alpha)100\%$.

4.3. Validation by Simulation.

The jackknife procedure may be validated, in an empirical sense, by sampling experiments or computer simulation in the following manner. First, an artificial batch or sample of data is obtained by drawing random numbers from postulated distributions for U, and for D. For example, $\{u_i\}$ and $\{d_i\}$ are independently sampled from the exponential distributions with means $\mu^{-1} = 100$, and $\lambda^{-1} = 1$, respectively. Second, the jackknife point estimate ((4-2-2) above) and confidence limits ((4-2-3) above) are computed. Since the values of $E[U]$ and $E[D]$ are known, so is the theoretical value of A, The jackknife confidence intervals can be checked for coverage: if $L_\alpha \leq A \leq H_\alpha$ then the particular interval <u>covers</u>, while otherwise (if $A < L_\alpha$ or $H_\alpha < A$) it does not cover. Finally, the above procedure can be repeated many times (say 1000) and the fraction of the repetitions which contain the true value of A are recorded. This fraction of the coverage should desirably be close to $(1 - \alpha)$. Also, the average length, and variance of length, of the confidence intervals obtained in repeated sampling can be recorded. The jackknife confidence limits procedure can be said to be <u>robust of validity</u> if the actual coverage is close to the nominal coverage, $1 - \alpha$, for a wide range of distributions for U and D. The procedure can be said to be <u>robust of efficiency</u> if the confidence limits tend to be short, i.e., if there is evidence that $E[H_\alpha] - E[L_\alpha]$ is comparable to the length of confidence intervals obtained when the underlying distributional families for U and D are known,

and the most efficient procedures for estimation pertinent to these families, are used. Without the evidence available from a very large data base, choice of specific distributional forms for U and D must be based on judgment. The following example situations seem to reflect the types of distributional behaviors that may occur.

(A). U is exponentially distributed, $E[U] = \lambda^{-1}$.
D is exponentially distributed, $E[D] = \mu^{-1}$.
Successive times to failure and repair times are independent. Note: This is the widely seen Markov model, is mathematically convenient, and may well be reasonably accurate under many circumstances.

(B). U is exponentially distributed. D is gamma distributed with shape parameter, k, greater than unity: $E[D] = \mu^{-1}$, $Var[D] = (\sqrt{k}\,\mu)^{-2}$.
Note: the gamma family with $k > 1$ qualitatively represents data that is more tightly grouped around its mean than is true of exponentially distributed data. The logarithmic-normal distribution also has the above general property, and has been used to represent repair times; see Gray and Schucany [9].

(C). U is exponentially distributed, $E[U] = \lambda^{-1}$.

D is gamma, with k integer (>1); U and the subsequent D positively correlated.

Note: Situations in which repair times following longer-than-average times to failure are themselves longer-than-average can be imagined. A class of models is discussed in Gaver [6]. The present simulation is a simplified version of such a structure.

(D). U is represented by a long-tailed h-distribution, see Gaver and Lavenburg [7], and Rogers and Tukey [13]:

$$U = \frac{(1-h)^2}{\lambda} X e^{hX} \qquad h>0$$

where X is exponentially distributed with unit mean. The distribution of U possesses exponential-like characteristics near zero, but exhibits relatively more extremely large times to failure than does the exponential.

D is exponential; $E[D] = \mu^{-1}$.

The above alternatives are by no means exhaustive, but do tend to represent qualitatively likely alternative data behaviors. As the following tabulations indicate, the jackknife appears to

Table 4.1

Simulation Experiments Validating Jackknife

Single-Unit Availability

95% Confidence Limits; Two-Sided ($t = 2.064$)

$n = 25$

	Underlying Distributions	Coverage (%)	Average Length	Variance Length
A.	(exponential)	JK: 96.2	1.27×10^{-2}	2×10^{-5}
	$\lambda = 0.01$, $\mu=1$	"F": 95.9	1.21×10^{-2}	1×10^{-5}
B.	(exponential, gamma)	JK: 94.2	9.88×10^{-3}	1×10^{-5}
	$\lambda = 0.01$, $\mu=1$	"F": 98.8	1.22×10^{-2}	1×10^{-5}
	$k=3$			
C.	(exponential, corr. gamma)	JK: 94.7	6.00×10^{-3}	0.4×10^{-5}
	$\lambda = 0.01$, $\mu=1$	"F": 99.9	1.20×10^{-2}	0.3×10^{-5}
	$k=2$			
D.	(long-tailed h, exponential)	JK: 94.1	1.64×10^{-2}	4×10^{-5}
	$\lambda = 0.01$, $\mu=1$	"F": 88.7	1.27×10^{-2}	2×10^{-5}
	$h=0.2$			

Table 4.2

Simulation Experiments Validating Jackknife

Single-Unit Availability

95% Confidence Limits; Two-Sided ($t = 2.145$)

$n = 15$

	Underlying Distributions	Coverage (%)	Average Length	Variance Length
A.	(exponential)	JK: 95.0	1.94×10^{-2}	1×10^{-4}
	$\lambda = 0.01$, $\mu=1$	"F": 94.4	1.63×10^{-2}	4×10^{-5}
B.	(exponential, gamma)	JK: 94.1	1.39×10^{-2}	3×10^{-5}
	$\lambda = 0.01$, $\mu=1$	"F": 97.7	1.64×10^{-2}	3×10^{-5}
	$k=3$			
C.	(exponential, corr. gamma)	JK: 92.2	8.4×10^{-3}	1.7×10^{-5}
	$\lambda = 0.01$, $\mu=1$	"F": 99.5	1.6×10^{-2}	0.98×10^{-5}
	$k=2$			
D.	(long-tailed h, exponential)	JK: 92.4	2.42×10^{-2}	1.8×10^{-4}
	$\lambda = 0.01$, $\mu=1$	"F": 88.0	1.75×10^{-2}	7×10^{-5}
	$h=0.2$			

perform creditably when data comes from any one of the models described. In particular, the validity of the jackknife is notable when a long-tailed (type D) distribution governs the times to failure.

In case (A) of Tables 4.1 and 4.2, the ratio $\frac{\bar{U}}{\bar{D}}$ is proportional to the F distribution of classical statistics, with degrees of freedom in numerator (denominator) equal to twice the number of up time (down time) observations. This fact allows exact confidence intervals to be established in case (A) -- and in case (A) alone --for any sample size. The jackknife coverage and confidence interval width compares favorably to the exact "F" method in case (A), and seems correspondingly more valid and efficient in the other cases considered. This is particularly true for the long-tailed distributions of type (D); here the "F" method considerably undercovers.

4.4. Numerical Applications

In order to illustrate the behavior of the jackknifed estimation procedure, consider system time to failure and time to repair data for two nuclear plants, as quoted by Tietjens and Waller [14]. The data are tabulated in Table 4.3.

For each set of data, the Jackknife pseudovalues are obtained by successively leaving out up and down time pairs, using equation (4-2-3). The two-sided confidence limits equation (4-2-7) are computed.

Table 4.3

$\{u_i\}$ and $\{d_i\}$ of Humboldt Bay
and Yankee Reactor [14]

Humboldt Bay		Yankee Nuclear	
Up Times (years)	Down Times (years)	Up Times	Down Times
0.523	0.060	0.063	0.027
0.175	0.038	0.055	0.038
0.537	0.074	0.296	0.014
1.019	0.197	0.170	0.036
0.121	0.016	0.822	0.345
0.827	0.088	0.948	0.197
0.271	0.016	0.715	0.096
0.499	0.066	0.923	0.255
0.940	0.058	0.899	0.090
0.466	0.099	0.332	0.033
0.742	0.060	0.304	0.049
0.189	0.058	0.658	0.107
0.422	0.016	0.523	0.019
0.389	0.222	0.712	0.148
1.000	0.118	0.485	0.022
0.003	0.047	0.397	0.030
0.855	0.085	0.145	0.101
1.077	0.153	0.912	0.019
		0.244	0.260

These confidence limits are compared to the limits obtained by Tietjens and Waller [14]. It is noticed in Table 4.4 that the jackknifed intervals fall within the F-statistic intervals, and also within the simulation intervals. As will appear from the simulation results of the following section, the jackknife procedure gives more uniformly valid confidence intervals than does the F procedure when the underlying distributions are not known. This robustness is a point in favor of the jackknife, from a practical viewpoint, for sampling experiments have confirmed its validity.

Table 4.4

Two-Sided 95% Confidence Limits on Plant Availability

		Lower Limit	Upper Limit
Yankee	Simulat. (Tietjens-Waller)	0.710	0.909
Nuclear:	Jackknife	0.762	0.887
(n=19)	F	0.729	0.906
Humboldt	Simulat. (Tietjens-Waller)	0.779	0.923
Bay	Jackknife	0.829	0.905
(n=18)	F	0.778	0.930

4.5. Jackknifing System Availability

The topic of this section is the estimation of the availability of a system of several (two or more) equipments from time to failure and repair data. Again the jackknife technique is emphasized. Variations of this method are described and are again evaluated by means of simulation.

4.5.1 Two specific, simple, systems will be considered here for illustration.

System Type 1. Two Component Redundant.

Two subsystems are arranged in parallel, so that in order for the entire system to fail, both must be down simultaneously. If A_i is the availability of the i^{th} $(i = 1,2)$ then the system unavailability is

$$\bar{A} = (1-A_1)(1-A_2) = \bar{A}_1 \bar{A}_2$$

$$= \left(\frac{E[D_1]}{E[U_1]+E[D_1]} \right) \left(\frac{E[D_2]}{E[U_2]+E[D_2]} \right) \qquad (4\text{-}4\text{-}1)$$

under the assumption that the two systems fail and are repaired independently. If there are K such subsystems, then of course

$$\bar{A} = \prod_{i=1}^{K} \frac{E[D_i]}{E[U_i]+E[D_1]}$$

System Type 2. Two-Out-of-Three Voting.

Suppose three subsystems are arranged to vote: when a demand is made for the system then if at least two out of three subsystems are available, the system is itself available. The system availability in terms of subsystem availability is given as below

$$A = A_1 A_2 A_3 + \bar{A}_1 A_2 A_3 + A_1 \bar{A}_2 A_3 + A_1 A_2 \bar{A}_3 \qquad (4\text{-}4\text{-}2)$$

4.5.2 Some Jackknife Procedures

If a system consists of subsystems which are assumed to be identical and independent then data on times to failure and times to repair can be pooled. The jackknife procedure discussed in Section 4.4 requires only a modest adaptation.

(A) Jackknifing System Type 1; Identical Subsystems.

Since subsystems behave identically, by assumption

$$E[D] = E[D_1] = E[D_2], \ E[U] = E[U_1] = E[U_2]$$

$$\bar{A}_j = \frac{E[D]}{E[U]+E[D]} \qquad (4\text{-}4\text{-}3)$$

and thus

$$\frac{\bar{A}^{\frac{1}{2}}}{1-\bar{A}^{\frac{1}{2}}} = \frac{E[D]}{E[U]} \qquad (4\text{-}4\text{-}4)$$

This suggests the following procedure

(1) Transform:

$$\ln \frac{\tilde{\bar{A}}^{\frac{1}{2}}}{1-\tilde{\bar{A}}^{\frac{1}{2}}} = \ln \bar{d} - \ln \bar{u} = -z \qquad (4\text{-}4\text{-}5)$$

(2) Jackknife z, in the manner described in Sec. 4.1, pooling all up time and down time data. The previously reported sampling experiments for one equipment indicate the validity of the intervals so obtained; two-sided confidence limits are of this form:

$$\left(\frac{1}{1+e^{H_\alpha}}\right)^2 \leq \bar{A} \leq \left(\frac{1}{1+e^{L_\alpha}}\right)^2 \qquad (4\text{-}4\text{-}6)$$

and other limits are found in an analogous manner.

(B) Jackknifing System Type 1; Different Subsystems.

It is often unrealistic to assume that redundant subsystems have identical parameters. In this case

$$\bar{A} = \bar{A}_1 \bar{A}_2 = \left(\frac{E[D_1]}{E[D_1]+E[U_1]}\right)\left(\frac{E[D_2]}{E[D_2]+E[U_2]}\right), \qquad (4\text{-}4\text{-}7)$$

and a logarithmic transformation is suggested:

$$\ln \bar{A} = \ln \bar{A}_1 + \ln \bar{A}_2;$$

it is this function that will be jackknifed. Let u_{ki} denote the $i^{\underline{th}}$ time to failure of equipment k $(k=1,2;\ i=1,2\ldots n_k)$,

and let d_{ki} be the corresponding down time. Here are two jackknife procedures.

<u>Procedure 1.</u>

(1) Compute the pseudovalues $z_{k,j}$, for each subsystem's data as described by equation (4-2-3).

(2) Compute the pseudovalues

$$\ell_{k,j} = \ell n\, \tilde{A}_{k,j} = -\ \ell n(1+e^{z_{k,j}});\ k=1,2,\quad j=1,2,\ldots,n_k \qquad (4\text{-}4\text{-}8)$$

(3) The means and variances of $\ell_{k,j}$ are given by

$$M_k = \frac{1}{n_k}\sum_{j=1}^{n_k} \ell_{k,j} \qquad (4\text{-}4\text{-}9)$$

$$S_k^2 = \frac{1}{n_k-1}\sum_{j=1}^{n_k}(\ell_{k,j}-M_k)^2 \qquad (4\text{-}4\text{-}10)$$

and

$$M = \sum_{j=1}^{2} M_j\ ,\quad S^2 = \frac{1}{n_1+n_2-2}\left[\sum_{j=1}^{n_1}(\ell_{1,j}-M_1)^2+\sum_{j=1}^{n_2}(\ell_{2j}-M_2)^2\right] \qquad (4\text{-}4\text{-}11)$$

(4) Two-sided $(1-\alpha)100\%$ confidence intervals of ℓnA are computed using Student's t:

$$H_\alpha = M + t_{1-\alpha/2}(n_1+n_2-2)\cdot S \qquad (4\text{-}4\text{-}12)$$

$$L_\alpha = M - t_{1-\alpha/2}(n_1+n_2-2)\cdot S$$

(5) Translated into confidence limits on A, the limits become

$$H_\alpha(A) = e^{H_\alpha}, \quad \text{and} \quad L_\alpha(A) = e^{L_\alpha} \qquad (4\text{-}4\text{-}13)$$

respectively; these limits are analogous to those of equation (4-2-2).

Note 1: Procedure 1 directly assesses the variability of the individual estimates of $\bar{A}_1$ and $\bar{A}_2$ in terms of functions of the original pseudovalues.

Note 2: The procedure is essentially equivalent to the statistical independent-t test applied to the jackknifed data.

Procedure 2.

An alternative approach is to compute the jackknife estimate of the (un)availability of each subsystem, and then to assess and combine the variabilities of these estimates.

(1) Compute the pseudovalues $z_{k,j}$ and the sample mean, m_k, and sample variance s_k of each subsystem's data.

(2) Calculate the logs of the jackknife point estimates,

$$M_k = \ln \tilde{\bar{A}}_{k,JK} = - \ln \left[1 + e^{\bar{z}_k} \right]; \quad k=1,2 \qquad (4\text{-}4\text{-}14)$$

$$M = M_1 + M_2.$$

(3) Compute the variance of the jackknife point estimates by using the asymptotic "linearization" or "small errors" approach [5],

$$\mathrm{Var}\left[\ln \tilde{\bar{A}}_{K,JK}\right] \approx \left(\frac{e^{\bar{z}_k}}{1+e^{\bar{z}_k}}\right)^2 \frac{s_k^2}{n_k} \quad k=1,2 \tag{4-4-15}$$

and the variance of point estimate A_{JK} is

$$S^2 = \sum_{k=1}^{2} \mathrm{Var}\left[\ln \tilde{\bar{A}}_{K,JK}\right] \tag{4-4-16}$$

(4) Construct the confidence limits of system unavailability in the same manner as equations (4-4-12) and (4-4-13).

(C) <u>Jackknifing System Type 2; Different Subsystems</u>.

The availability of a two-out-of-three voting system when components differ is given by equation (4-4-2). Suppose that up and down time data are known for the components, this section describes a jackknife procedure for applying confidence limits to the system availability. The method given here relies upon the linearization technique used as the basis for Procedure 2 of (B).

Procedure:

(1) Form the pseudovalues for the jackknife estimates of

$$\ln(E[u_k]/E[D_k]) \quad z_{k,j}, \quad k=1,2,3;\ j=1,2,\ldots n_k\ . \tag{4-4-17}$$

(2) Compute

$$\bar{z}_k = \frac{1}{n_k} \sum_{j=1}^{n_k} z_{k,j} \quad , \tag{4-4-18}$$

$$s_k^2 = \frac{1}{n_k - 1} \sum_{j=1}^{n_k} (z_{k,j} - \bar{z})^2 \qquad k=1,2,3 \ .$$

(3) Compute the jackknife point estimate of system availability

$$A_{JK} = A_{1,JK}A_{2,JK}A_{3,JK} + \bar{A}_{1,JK}A_{2,JK}A_{3,JK} + A_{1,JK}\bar{A}_{2,JK}A_{3,JK} + A_{1,JK}A_{2,JK}\bar{A}_{3,JK} \tag{4-4-19}$$

and its log-odds transform

$$\ell_{JK} = \ln \frac{A_{JK}}{1 - A_{JK}} \tag{4-4-20}$$

(4) Compute the estimated variance of ℓ_{JK}:

$$\begin{aligned} s_\ell^2 = \frac{1}{(A_{JK}\bar{A}_{JK})^2} \quad & [\bar{A}_{2,JK}A_{3,JK} + A_{2,JK}\bar{A}_{3,JK}]^2 [A_{1,JK}\bar{A}_{1,JK}]^2 \frac{s_1^2}{n_1} + \\ & [A_{1,JK}\bar{A}_{3,JK} + \bar{A}_{1,JK}A_{3,JK}]^2 [A_{2,JK}\bar{A}_{2,JK}]^2 \frac{s_2^2}{n_2} + \\ & [A_{1,JK}\bar{A}_{2,JK} + \bar{A}_{1,JK}A_{2,JK}]^2 [A_{3,JK}\bar{A}_{3,JK}]^2 \frac{s_3^2}{n_3} \ ; \end{aligned} \tag{4-4-21}$$

the latter is derived by linearizing equations (4-4-18) and (4-4-19) and combining.

(5) Two-sided $(1-\alpha)100\%$ confidence limits for $\ell n \frac{A}{1-A}$ are

$$H_\alpha = \ell_{JK} + t_{1-\alpha/2}(n_1+n_2+n_3-3).s_\ell$$

$$L_\alpha = \ell_{JK} - t_{1-\alpha/2}(n_1+n_2+n_3-3).s_\ell \; ; \qquad (4\text{-}4\text{-}22)$$

two-sided confidence limits on A are given by equation (4-2-8).

4.6 Validation by Simulation.

Sampling experiments designed to validate the procedures described do so in a satisfactory manner for the cases considered. The following tables illustrate the situation. Note that there is no "exact" finite-sample procedure analogous to use of the "F" statistic available for the single-unit situations when distributions are assumed to be exponential. Further sampling experiments, unreported here, also show that the nominal coverage is very nearly achieved in all cases.

Table 3.1

Simulation Experiments Validating Jackknife

Two-Unit Redundant Unavailability

95% Confidence Limits

$n = 25$

(JK, 1: Jackknife Procedure 1; JK-2: Jackknife Procedure 2)

	Underlying Distributions	Coverage (%)	Average Length	Variance Length
A.	(exponential up, expon. down)	JK,1: 93.9	3.3×10^{-4}	1.7×10^{-8}
	$\lambda_1 = 0.01$, $\lambda_2 = 0.02$	JK,2: 94.7	3.7×10^{-4}	2.4×10^{-8}
	$\mu_1 = 1 - \mu_2$			
B.	(exponential up, gamma down)	JK,1: 94.5	2.8×10^{-4}	1×10^{-8}
	$\lambda_1 = 0.01$, $\lambda_2 = 0.02$	JK,2: 94.6	2.8×10^{-4}	1.1×10^{-8}
	$\mu_1 = \mu_2 = 1$; shape = 3			
D.	(long-tailed h up, exponential down)	JK,1: 93.0	4.6×10^{-4}	4.5×10^{-8}
	$\lambda_1 = 0.01$, $\lambda_2 = 0.02$	JK,2: 93.0	5.1×10^{-4}	6.0×10^{-8}
	$\mu_1 = \mu_2 = 1$, $h = 0.2$			

Table 3.2

Simulation Experiments Validating Jackknife

Two-Out-of-Three Voting System

95% Confidence Limits

$n_1 = n_2 = n_3 = 15$

	Underlying Distributions	Coverage %	Average Length	Variance Length
A.	(exponential up, expon. down)	94.5	$3.0\text{x}10^{-3}$	$2.1\text{x}10^{-6}$
	$\lambda_1 = 0.01$, λ_2 0.02,			
	$\mu_1 = \mu_2 = \mu_3 = 1$			
B.	(exponential up, gamma down)	94.2	$2.2\text{x}10^{-3}$	$8.5\text{x}10^{-7}$
	$\lambda_1 = 0.01$, λ_2 0.02			
	λ_3 0.04,			
	$\mu_1 = \mu_2 = \mu_2$ 1			
D.	(longtailed h up,	92.9	$4\text{x}10^{-3}$	$4.8\text{x}10^{-6}$
	$\lambda_1 = 0.01$, $\lambda_2 = 0.02$,			
	λ_3 0.04,			
	$\mu_1 = \mu_2 = \mu_3 = 1$			

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